Shake slice knots

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Based on joint work with:

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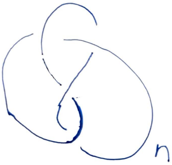
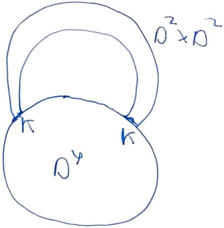
A *knot* is a submanifold of S^3 that is homeomorphic to S^1.

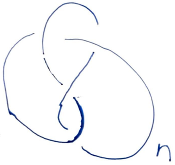
Let K be a knot and let $n \in \mathbb{Z}$. The *n-trace* $X_n(K)$ is the 4-manifold

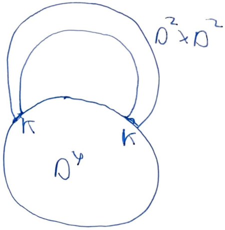
$$X_n(K) = D^4 \cup_{\nu K \sim S^1 \times D^2} D^2 \times D^2$$

with $S^1 \times D^2$ identified with an n-framed regular neighbourhood of $K \subset S^3 = \partial D^4$.









Note that

- $X_n(K)$ is a compact, simply connected 4-manifold;
- $X_n(K) \simeq S^2$; and
- $\partial X_n(K) = S^3_n(K)$, the result of n-framed surgery on S^3 along K.

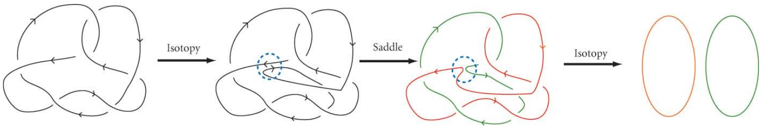
Question

For which K and which n is a generator of $\pi_2(X_n(K)) \cong \mathbb{Z}$ represented by a locally flat embedded 2-sphere?

Given a knot K and $n \in \mathbb{Z}$ for which this question has an affirmative answer, we say that K is *n-shake slice*.

Equivalently, some "n-shaking" of K bounds a locally flat planar surface in D^4.

A knot K in S^3 is *slice* if there is a locally flat embedded disc $D^2 \hookrightarrow D^4$ with boundary K.



Knots form a monoid under connected sum, knots modulo slice knots forms a group, the knot concordance group.

Partial Answer

If K is a slice knot, it is n-shake slice for every $n \in \mathbb{Z}$.

On the other hand, corollaries to the theorem that I will explain include:

Corollary (FMNOPR)

For every $n \neq 0$ there exist infinitely many knots that are n-shake slice but not slice. These knots may be chosen to be distinct in concordance.

It is unknown whether every 0-shake slice knot is slice.

Moreover the answer depends on n.

Corollary (FMNOPR)

If m does not divide n then there exist infinitely many knots which are n-shake slice but not m-shake slice. These knots may be chosen to be distinct in concordance.

I want to discuss a refinement of the question that I can answer.

Question

For which K and which n is a generator of $\pi_2(X_n(K)) \cong \mathbb{Z}$ represented by a locally flat embedded 2-sphere S such that the complement $X_n(K) \setminus S$ has abelian fundamental group?

Such a K is called $\mathbb{Z}/n$*-shake slice*.

Note in this case $\pi_1(X_n(K) \setminus S) \cong \mathbb{Z}/n$.

To explain the answer, let me recall some elementary knot invariants.

A *Seifert surface* for a knot K is a submanifold of S^3 homeomorphic to a connected, compact, orientable surface with one boundary component, such that the boundary equals K.

Theorem (Seifert)

Every knot bounds a Seifert surface.

Given a Seifert surface F for a knot K, the *Seifert form*

$$V\colon H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Z}$$

can be computed on homology classes represented by simple closed curves $\gamma, \delta \subset F$ by the linking number

$$\ell k(\gamma, \delta^+)$$

where δ^+ is a push-off of δ normal to F.

Can obtain knot invariants from V. Represent V by a $2g \times 2g$ matrix A.

1. The *Alexander polynomial* is

$$\Delta_K(t) := \det(tA - A^T) \in \mathbb{Z}[t, t^{-1}].$$

2. The Levine-Tristram signature at $z \in S^1$ is

$$\operatorname{sign}\left((1-z)A + (\overline{1-z})A^T\right) \in \mathbb{Z}.$$

3. The *Arf invariant* $\operatorname{Arf}(K) \in \mathbb{Z}/2$ is determined by

$$\Delta_K(-1) = \det(A + A^T) \equiv \pm(1 + 4\,\operatorname{Arf}(K)) \mod 8.$$

A knot is $\mathbb{Z}$-slice if it is slice via a locally flat disc D with $\pi_1(D^4 \setminus D) \cong \mathbb{Z}$.

Theorem (Freedman-Quinn)

A knot K is $\mathbb{Z}$-slice if and only if $\Delta_K(t) = 1$.

This extends fairly easily to a 0-shake slice analogue.

Theorem (Folklore, FNMOPR)

A knot K is $\mathbb{Z}$-shake slice if and only if $\Delta_K(t) = 1$.

Is there a version of this result for $n \neq 0$?

Theorem (Boyer, FNMOPR)

A knot K is 1-shake slice if and only if $\mathrm{Arf}(K) = 0$.

Proof.

- $S^3_1(K)$ is a $\mathbb{Z}HS^3$ so bounds a contractible topological 4-manifold W (Freedman).
- Interior connected sum $X := X\#\mathbb{CP}^2$ satisfies $\pi_1(X) = \{1\}$, $\partial X = S^3_1(K)$ and $Q_X = \langle 1 \rangle$.
- Boyer classified simply connected 4-manifolds with a fixed boundary: $X \cong X_1(K)$ if and only if $0 = \mathrm{ks}(X) = \mu(S^3_1(K)) = \mathrm{Arf}(K)$ (Gonzalez-Acuña).
- Image of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ is the desired embedded sphere in $X_1(K)$.

$\square$

The main theorem of this talk is a generalisation to all $n \in \mathbb{Z}$.

Theorem (FMNOPR)

A knot $K \subseteq S^3$ is $\mathbb{Z}/n$-shake slice if and only if:

(i) $H_1(S^3_n(K); \mathbb{Z}[\mathbb{Z}/n]) = 0$*; or equivalently for* $n \neq 0$*,*

$$\prod_{\{\xi \mid \xi^n = 1\}} \Delta_K(\xi) = 1.$$

(ii) $\mathrm{Arf}(K) = 0$*; and*

(iii) $\sigma_K(\xi) = 0$ *for every* $\xi \in S^1$ *such that* $\xi^n = 1$*.*

Moreover, any two $\mathbb{Z}/n$-shake slicing spheres are ambiently isotopic rel. boundary in $X_n(K)$.

Let $D^2 \to D_n \to S^2$ be the fibre bundle with Euler number n.

Proposition

A knot K is $\mathbb{Z}/n$-shake slice if and only if $S^3_n(K)$ is homology cobordant to $L(n,1)$ via a cobordism V with $\pi_1(V) \cong \mathbb{Z}/n$, such that $V \cup D_n \cong X_n(K)$.

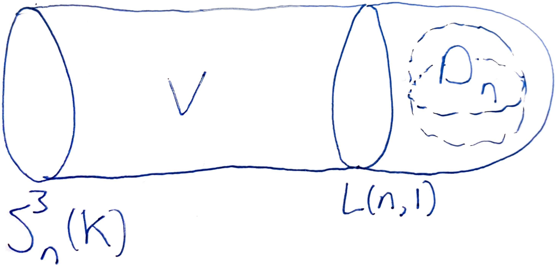
Proof.

If K is $\mathbb{Z}/n$-shake slice then the exterior of the embedded 2-sphere is such a homology cobordism V.

If such a V exists, the image in $X_n(K)$ of the zero section of D_n is a locally flat 2-sphere as required. □

Proposition

A knot K is $\mathbb{Z}/n$-shake slice if and only if $S^3_n(K)$ is homology cobordant to $L(n,1)$ via a cobordism V with $\pi_1(V) \cong \mathbb{Z}/n$, such that $V \cup D_n \cong X_n(K)$.



Proof.

Steps in the proof of main theorem, only if direction:

- Suppose $(V; S^3_n(K), L(n,1))$ exists as in the Proposition.
- Homology computation with V shows that $H_1(S^3_n(K); \mathbb{Z}[\mathbb{Z}/n]) = 0$.
- Let $\xi = e^{2\pi i/n}$. Use G-signature theorem to show

$$\sigma_K(\xi^k) = \operatorname{sign}(Q(\widetilde{W})_k) - \operatorname{sign}(Q_W),$$

where $Q(\widetilde{W})_k$ is intersection pairing of $\mathbb{Z}/n$-cover of cobordism $(W^4; S^3_n(K), L(n,1))$, restricted to ξ^k-eigenspace: use $W = V$ to see $\sigma_K(\xi^k) = 0$.

$\square$

Proof.

Steps in the proof of main theorem, only if direction:

- The Arf invariant obstructs the existence of a bordism $(V; S^3_n(K), L(n,1))$ over $\mathbb{Z}/n$ for n even, so $\mathrm{Arf}(K) = 0$.
- The Arf invariant of K can be identified with the Kirby-Siebenmann invariant of $V \cup D_n$ when is n odd. $V \cup D_n \cong X_n(K)$ smooth implies $\mathrm{ks}(X_n(K)) = 0$.

$\square$

Proof.

Steps in the proof of main theorem, if direction:

- Use bordism theory and $\mathrm{Arf}(K) = 0$ (n even) to construct a degree one normal bordism

$$F\colon (W; S^3_n(K), L(n,1)) \to (L(n,1) \times I; L(n,1), L(n,1)).$$

- $H_1(S^3_n(K); \mathbb{Z}[\mathbb{Z}/n]) = 0$ implies $Q(\widetilde{W})$ determined an element of $L^s_4(\mathbb{Z}[\mathbb{Z}/n])$, simple surgery obstruction group $=$ Witt group of quadratic forms.

- $L^s_4(\mathbb{Z}[\mathbb{Z}/n]) \cong \mathbb{Z}^{r(n)}$ where

$$r(n) = \begin{cases} (n+1)/2 & n \text{ odd} \\ (n+2)/2 & n \text{ even.} \end{cases}$$

These $\mathbb{Z}$ summands are identified with $\sigma_{\xi^k}(K)$, so vanish.

$\square$

Proof.

Steps in the proof of main theorem, if direction:

- Perform surgery on W: $\mathbb{Z}/n$ is a good group, represent $\ker \pi_2(F)$ by locally flat embedded S^2s. $\mathrm{cl}(W \setminus S^2 \times D^2) \cup D^3 \times S^1$.
- Obtain homology cobordism $(V; S^3_n(K), L(n,1))$ with $\pi_1(V) \cong \mathbb{Z}/n$.
- $V \cup D_n \cong X_n(K)$ by Boyer's classification if n even, and the same holds for n odd if and only if $\mathrm{ks}(V \cup D_n) = 0$. But $\mathrm{ks}(V \cup D_n) = \mathrm{Arf}(K) = 0$.
- Zero section of D_n maps to locally flat embedded S^2 as required.

$\square$

Corollary (FMNOPR)

For every $n \neq 0$ there exist infinitely many knots that are n-shake slice but not slice. These knots may be chosen to be distinct in concordance.

Proof.

The knots $K_{n,j} := C_{n,1}(T_{2,8j+1})$ are all $\mathbb{Z}/n$-shake slice by our theorem, since for every ω with $\omega^n = 1$:

$$\sigma_\omega(C_{n,1}(J_j)) = \sigma_{\omega^n}(J_j) = \sigma_1(J_j) = 0$$

$$\Delta_{C_{n,1}(J_j)}(\omega) = \Delta_{J_j}(\omega^n) = \Delta_{J_j}(1) = 1$$

Also $\mathrm{Arf}(C_{n,1}(J_j)) = n\,\mathrm{Arf}(J_j)$.

With $J_j = T_{2,8j+1}$, $\mathrm{Arf}(J_j) = 0$ so $\mathbb{Z}/n$-shake slice but signature functions are distinct, so non-slice and mutually non-concordant. $\square$

Corollary (FMNOPR)

If m does not divide n then there exist infinitely many knots which are n-shake slice but not m-shake slice. These knots may be chosen to be distinct in concordance.

Proof.

The knots $J_j := C_{n,1}(T_{2,8(N+j)+1})$ work, $j \geq 0$, for N sufficiently large so that $\sigma_{T_{2,8N+1}}(e^{2\pi ik/q}) \neq 0$ for q a prime power dividing m but not n, and $k = 1, \ldots, q-1$.

$\square$

On the other hand:

Corollary (FMNOPR)

If $m \mid n$ and K is $\mathbb{Z}/n$-shake slice, then K is $\mathbb{Z}/m$-shake slice.

Some more corollaries:

Corollary (FMNOPR)

For every $n \neq 0$ there exist infinitely many topological concordance classes of knots that are n-shake slice but not smoothly n-shake slice.

Corollary (FMNOPR)

There exist knots that are $\mathbb{Z}/n$-shake slice for infinitely many $n \in \mathbb{Z}$, but are not slice.

Corollary (FMNOPR)

For each odd $n \in \mathbb{N}$, there exists K such that $S^3_n(K)$ and $S^3_n(U)$ are topologically homology cobordant but K is not n-shake slice.

Corollary

A knot is (± 1)*-shake slice if and only if it is* $\mathbb{Z}/1$*-shake slice. For all other n, there exists an n-shake slice knot that is not* $\mathbb{Z}/n$*-shake slice.*

Corollary

If a knot K is n-shake slice for some integer n, then it is (± 1)*-shake slice.*